

On Virtual Crossing Number Estimates For Virtual Links

Denis Afanasiev* and Vassily Manturov

Abstract

We address the question of detecting minimal virtual diagrams with respect to the number of virtual crossings. This problem is closely connected to the problem of detecting the minimal number of additional intersection points for a generic immersion of a singular link in R^2 .

We tackle this problem by the so-called ξ -polynomial whose leading (lowest) degree naturally estimates the virtual crossing number.

Several sufficient conditions for minimality together with infinite series of new examples are given.

We also state several open questions about M -diagrams, which are minimal according to our sufficient conditions.

1 Introduction

One of the most important problems in the classification of knots and links is the problem of detecting the minimal crossing number. One of the striking achievements in that direction is the celebrated Kauffman-Murasugi theorem [Mur, Thi], for further generalizations see [Ma3] establishing the minimality of reduced alternating diagrams for virtual links. This theorem (conjectured by Tait in late 19-th century) was solved in late 1980-s by using a newborn polynomial: the Kauffman bracket version [Ka2] of the Jones polynomial [Jon].

In mid-90-s, Louis Kauffman introduced [Ka1] a natural generalization of knots and links called virtual knot theory. This generalization can be treated both topologically (as knots in thickened 2-surfaces up to homotopy and stabilization/destabilization) and combinatorially (via a generalization of planar diagrams with a new crossing type — called virtual — allowed). Virtual knot theory played a crucial role in solving some problems about classical knots (see,

*Corresponding author

e.g., [GPV]) and for understanding of some notions in classical knot theory, and stating new ones (see, e.g.[FKM]).

One of the most natural problems in virtual knot theory is to estimate classical and virtual crossing numbers. For estimating classical crossing number, a lot of results were obtained (see, e.g., [Ma2, Miy, Ma3] and references therein) by using some generalization of the Kauffman-Murasugi theorem etc.

The problem this present paper is devoted to is to estimate from below the virtual crossing number: a partial case of this theorem detects non-classicality of a link (in this direction we note the paper [DK]). Note that this problem is closely connected to the problem of estimating minimal number of crossings for projecting a given singular link to a plane.

We attack the virtual crossing number by using the ξ -polynomial introduced independently by several authors (see [KR],[Saw],[SW],[Ma1], for the proof of their coincidence, see [BF], for detecting non-classicality see also [Tep]), and we use the definition by the second named author of the present paper: this definition includes a “counting variable” for virtual crossings, and it is natural that the degrees of the polynomial invariant in this counting variable estimates the virtual crossing number from below.

The paper is organized as follows.

The next section contains all basic notions and the main Theorem (Theorem 1) on virtual crossing estimates.

In section 3, we prove some theorems delivering sufficient conditions for minimality and present several series of examples of minimal diagrams.

We conclude our paper by section 4 with a list of unsolved problems.

1.1 Acknowledgements

The authors are grateful to L.H.Kauffman, A.T.Fomenko and V.A.Vassiliev for their attention to this work and fruitful consultations.

2 Basic Notions and Constructions

Virtual knot theory was invented by Kauffman around 1996, [Ka1].

Definition 2.1. A *virtual diagram* is a 4-valent diagram in \mathbb{R}^2 where each crossing is either endowed with a classical crossing structure (with a choice for underpass and overpass specified) or just said to be virtual and marked by a circle.

Definition 2.2. A *virtual link* is an equivalence class of virtual link diagram modulo generalised Reidemeister moves. The latter consist of usual Reidemeister moves referring to classical crossings and the *detour move* that replaces one

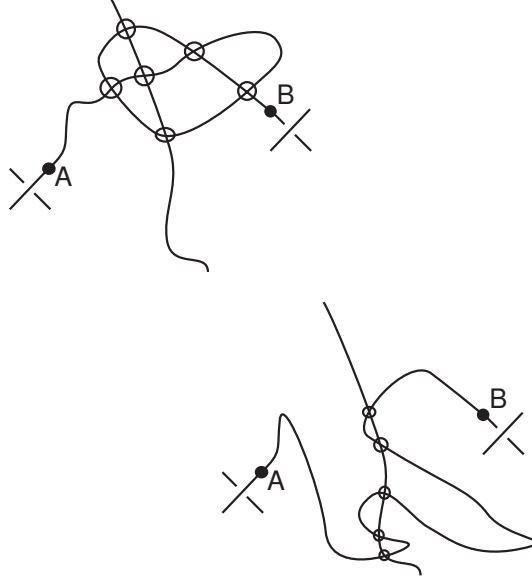


Figure 1: The detour move

arc containing only virtual intersections and self-intersection by another arc of such sort in any other place of the plane, see Fig. 1.

Throughout the paper, we deal only with oriented links. By an *arc* of a planar diagram we mean a connected component of the set, obtained from the diagram by deleting all virtual crossings (at classical crossing the undercrossing pair of edges of the diagram is thought to be disjoint as it is usually illustrated). We say that two arcs a, a' belong to the same *long arc* if there exists a sequence of arcs $a = a_0, \dots, a_n = a'$ and virtual crossings c_1, \dots, c_n such that for $i = 1, \dots, n$ the arcs a_i, a_{i+1} are incident to c_i from opposite sides. A planar virtual diagram is said to be *proper*, if it has no cyclic long arcs. It is easy to show that equivalent proper diagrams can be transformed to each other by using generalized Reidemeister moves in the class of proper diagrams. This can be done by adding/removing a curl (by using the first classical Reidemeister move) when necessary. For each proper diagram, the number of long arcs equals the number of classical crossings. In the sequel, we deal only with proper diagrams.

Now, let us construct the ξ -polynomial of virtual links (and its unnormalized version called ζ).

Let D be a proper diagram of a virtual link D with n classical crossings. Let us construct an $n \times n$ -matrix $A(D)$ with elements from $\mathbb{Z}[t, s, t^{-1}, s^{-1}]$ as follows.

First, we enumerate all classical crossings of D by integer numbers from 1 to n and associate with each crossing the emanating long arc. Each long arc

starts with a (short) arc. Let us associate the label 1 with latter arc. All other arcs of the long arc will be marked by exponents $s^k, k \in \mathbb{Z}$, as follows. While passing through the virtual crossing, we multiply the label by s if we pass from the left to the right or by s^{-1} otherwise.

Since the diagram is proper, our labeling is well defined. Consider a classical crossing v_i with number i . It is incident to some three arcs p, q, r , belonging to long arcs with numbers i, j, k , whence number j belongs to the arc passing through v_i . Denote the power of s of the label corresponding to q by a_{ij} , and that of the label corresponding to r by a_{ik} .

Let us define the i -th row of the matrix $A(D)$ as the sum of the following three rows y_1, y_2, y_3 of length n . Each of these rows has only one nonzero element. The i -th element of the row y_1 equals 1. If the crossing is positive, we set

$$y_{2k} = -s^{a_{ik}}t, y_{3j} = (t-1)s^{a_{ij}}; \quad (1)$$

otherwise we put

$$y_{2k} = -s^{a_{ik}}t^{-1}, y_{3j} = (t^{-1}-1)s^{a_{ij}} \quad (2)$$

We set $\zeta(D) := \det A(D)$. In [Ma1] it is proved that if two virtual diagrams D, D' are equivalent then $\zeta(D) = t^l \zeta(D')$ for some integer l .

Denote by $\deg f$ and $mdeg f$ the leading (lowest) degree of monomials $f \in \mathbb{Z}[t, t^{-1}][s, s^{-1}]$ with respect to variable s ; if $f = 0$, we set $\deg f = -\infty$, $mdeg f = +\infty$.

Theorem 2.1. *Let k be the number of virtual crossings of a virtual diagram D . Then $\deg \zeta(D) \leq k$, $mdeg \zeta(D) \geq -k$.*

Remark 2.1. *$\deg \zeta(D)$ and $mdeg \zeta(D)$ are virtual link invariants.*

Proof. Let v_1, \dots, v_n be an arbitrarily enumerated classical crossings of a virtual diagram D , $\gamma^1, \dots, \gamma^n$ be long arcs of D enumerated according to the rule: γ^i is emanating from v_i . Here k_i denotes number of virtual crossing that belong to γ^i , $\gamma_0^i, \dots, \gamma_{k_i}^i$ denote arcs on γ^i enumerated in their sequence on the long arc. Now, using this notation, we can get another definition of Alexander-like matrix $A = A(D)$.

The polynomial $[v_i : \gamma_\mu^j] \in \mathbb{Z}[t, t^{-1}]$, defined in following way, is called an *incidence coefficient* of classical crossing v_i and arc γ_μ^j . If γ_μ^j and v_i are not incident, we set $[v_i : \gamma_\mu^j] = 0$. If they are incident, we have the following options: the arc (1) is emanating from the crossing, (2) or passing through it, (3) or coming into it, or simultaneously realize some of conditions (1)-(3); accordingly, $[v_i : \gamma_\mu^j]$ is defined as a sum of some of three polynomials:

- (1) 1;
- (2) $t^{\text{sgn } v_i} - 1$;
- (3) $-t^{\text{sgn } v_i}$;

where $\text{sgn } v_i$ denotes a *local writhe number* of crossing v_i (Recall that the *local writhe number* of a classical crossing is +1 for $\diagup \diagdown$ and -1 for $\diagdown \diagup$. The *writhe number* of a (virtual) diagram is the sum of local writhe numbers over all classical crossings).

Thus, we set $A_{ij} := \sum_{\mu=0}^{k_j} [v_i : \gamma_\mu^j] s^{\deg \gamma_\mu^j}$, where $\deg \gamma_\mu^j$ denotes the power of variable s on arc γ_μ^j .

If we have a long arc passing through a virtual crossing, then this passing may be either *increasing* or *decreasing* (with respect to the power of s on the arcs of the long arc separated by the virtual crossing in question. More precisely, we say that a virtual crossing v is *increasing* on a long arc γ between two consecutive arcs γ_i and γ_{i+1} if the other long arc passing through v goes from the left to the right with respect to the orientation γ_i and γ_{i+1} . If the other long arc goes from the left to the right, we call this arc *decreasing*.

We note that each virtual crossing is increasing for exactly one long arc and decreasing for exactly one arc. These two arcs may coincide if a long arc passes twice through a virtual crossing.

Let p_j and q_j , $j = 1, \dots, n$, be numbers of increasing and decreasing virtual crossings on γ^j , respectively. Then $\max_{0 \leq \mu \leq k_j} \deg \gamma_\mu^j \leq p_j$ and $\min_{0 \leq \mu \leq k_j} \deg \gamma_\mu^j \geq -q_j$. Consequently, $\deg A_{ij} = \deg(\sum_{\mu=0}^{k_j} [v_i : \gamma_\mu^j] s^{\deg \gamma_\mu^j}) \leq \max_{0 \leq \mu \leq k_j} \deg \gamma_\mu^j \leq p_j$ and $mdeg A_{ij} = mdeg(\sum_{\mu=0}^{k_j} [v_i : \gamma_\mu^j] s^{\deg \gamma_\mu^j}) \geq \min_{0 \leq \mu \leq k_j} \deg \gamma_\mu^j \geq -q_j$. So, we have inequalities:

$$\deg A_{ij} \leq \max_{0 \leq \mu \leq k_j} \deg \gamma_\mu^j \leq p_j, mdeg A_{ij} \geq -q_j. \quad (3)$$

Hence, $\deg \zeta(D) = \deg(\sum_{\sigma \in S_n} (-1)^\sigma A_{\sigma_1 1} A_{\sigma_2 2} \dots A_{\sigma_n n}) \leq \max_{\sigma \in S_n} (\deg A_{\sigma_1 1} + \dots + \deg A_{\sigma_n n}) \leq p_1 + \dots + p_n$. Analogously, $mdeg \zeta(D) \geq \min_{\sigma \in S_n} (mdeg A_{\sigma_1 1} + \dots + mdeg A_{\sigma_n n}) \geq -q_1 - \dots - q_n$.

Since every virtual crossing is increasing for exactly one long arc and decreasing for exactly one long arc (possibly, the same), we get

$$p_1 + \dots + p_n = k, q_1 + \dots + q_n = k. \quad (4)$$

Thus, $\deg \zeta(D) \leq p_1 + \dots + p_n = k$ and $mdeg \zeta(D) \geq -q_1 - \dots - q_n = -k$. The theorem is proved. \square

3 Sufficient conditions for minimality

Assume for a proper virtual diagram D we have $\deg \zeta(D) = k(D)$. Then for every virtual diagram D' which equivalent to D we have $\zeta(D') = t^l \zeta(D) \Rightarrow$

$k(D) = \deg \zeta(D) = \deg \zeta(D') \leq k(D')$ by Theorem 2.1. Thus, D has minimal possible number of virtual crossings.

So, sufficient conditions for equality $\deg \zeta(D) = k(D)$ to hold are also sufficient for minimality of D with respect to the number of virtual crossings.

Now, our aim is to find necessary and sufficient conditions, when the equation $\deg \zeta(D) = k(D)$ holds. Let A_{ij}^l , $l \in \mathbb{Z}$, be the coefficients of the polynomial $A_{ij} \in \mathbb{Z}[t, t^{-1}][s, s^{-1}]$ defined as follows. $A_{ij} = \sum_{l=-\infty}^{+\infty} A_{ij}^l s^l$. We shall use the notation $(f)_k$ for the k -th coefficient of $f \in R[x, x^{-1}]$. Then, $(\zeta(D))_k = (\det A(D))_k = (\sum_{\sigma \in S_n} (-1)^\sigma A_{\sigma_1 1} A_{\sigma_2 2} \dots A_{\sigma_n n})_k = \sum_{\sigma \in S_n} (-1)^\sigma \sum_{l_1, \dots, l_n: l_1 + \dots + l_n = k} A_{\sigma_1 1}^{l_1} \dots A_{\sigma_n n}^{l_n} = \sum_{l_1, \dots, l_n: l_1 + \dots + l_n = k} \sum_{\sigma \in S_n} (-1)^\sigma A_{\sigma_1 1}^{l_1} \dots A_{\sigma_n n}^{l_n} = \sum_{\sigma \in S_n} (-1)^\sigma A_{\sigma_1 1}^{p_1} \dots A_{\sigma_n n}^{p_n}$, because $A_{ij}^l = 0$ for $l > p_j$ by (3) and $p_1 + \dots + p_n = k$ by (4). So, $(\zeta(D))_k = \det T$, where $T_{ij} := A_{ij}^{p_j}$.

Thus, to reach the maximal possible degree (minimal possible degree) of the ξ -polynomial, we need T to be non-degenerate, which, in turn, is possible only when each long arc contributes its maximal possible (minimal possible) degree of s to a prefixed summand of the determinant of T . More exactly, we have the following.

$$\deg \zeta(D) = k(D) \Leftrightarrow \det T(D) \neq 0. \quad (5)$$

The matrix $T = T(D)$ satisfies following

Statement 3.1. 1) $\det T(D) \neq 0 \implies \forall j \max_{0 \leq \mu \leq k_j} \deg \gamma_\mu^j = p_j$;
2) $\forall j \max_{0 \leq \mu \leq k_j} \deg \gamma_\mu^j = p_j \implies T_{ij}(D) = [v_i : \gamma_{p_j}^j]$.

Proof. (1) Let us suppose that $\exists j \max_{0 \leq \mu \leq k_j} \deg \gamma_\mu^j < p_j$. Then $\deg A_{ij} = \deg(\sum_{\mu=0}^{k_j} [v_i : \gamma_\mu^j] s^{\deg \gamma_\mu^j}) \leq \max_{0 \leq \mu \leq k_j} \deg \gamma_\mu^j < p_j \implies \forall i \deg A_{ij} < p_j \implies \forall i T_{ij} = A_{ij}^{p_j} = 0 \implies \det T(D) = 0$.

(2) $\max_{0 \leq \mu \leq k_j} \deg \gamma_\mu^j = p_j \implies \forall j \deg \gamma_{p_j}^j = p_j$ and $\deg \gamma_\mu^j < p_j$ for $\mu \neq p_j \implies T_{ij} = A_{ij}^{p_j} = (\sum_{\mu=0}^{k_j} [v_i : \gamma_\mu^j] s^{\deg \gamma_\mu^j})_{p_j} = [v_i : \gamma_{p_j}^j]$. \square

Statement 3.1 naturally leads to the following

Definition 3.1. (1) An arc $\gamma_{p_j}^j$, $j = 1, \dots, n$, $\deg \gamma_{p_j}^j = p_j$, we called *critical arc*. It is an arc which has a greatest possible degree.

(2) Virtual diagram D is called *special*, if every long arc of D contains a critical arc, i.e. $\forall j \max_{0 \leq \mu \leq k_j} \deg \gamma_\mu^j = p_j$. In other words, increasing virtual crossings on each long arc are located in the beginning of this long arc.

(3) For a special diagram D , the $n \times n$ -matrix $T = T(D) \in \mathbb{Z}[t, t^{-1}]$, $T_{ij} = [v_i : \gamma_{p_j}^j]$, is called the *T-matrix of special diagram D*. It is composed of incidence coefficients of classical crossings and critical arcs.

(4) We call a special diagram D with a nondegenerate T -matrix a *T-diagram*.

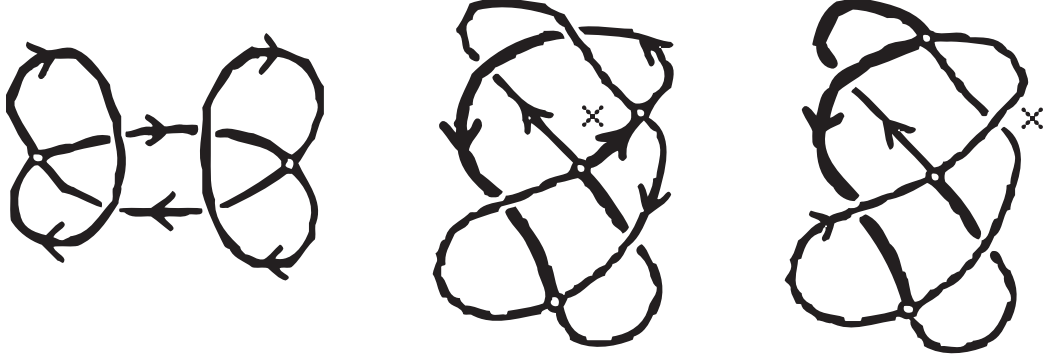


Figure 2: Special diagrams with $\deg \zeta < k$

From the definition above, statement 3.1 and relation (5), we easily conclude

Statement 3.2. *A proper virtual diagram is T -diagram if and only if $\deg \zeta(D) = k(D)$. Besides, for a special virtual diagram D the k -th coefficient of polynomial $\zeta = \zeta(D)$ is equal to determinant of T -matrix of this special diagram.*

Hence, if D is not special or it is special, but there is a critical arc which is not incident with some of classical crossing, or there is a classical crossing which is not incident some of critical arc, then $\deg \zeta(D) < k$. These options are illustrated in Figure 2.

Besides, by using Statement 3.2, we formulate an easier sufficient condition of minimality than come from Statement 3.2. Let us introduce following

Definition 3.2. For a special diagram D , we call the $n \times n$ -matrix $M = M(D)$, which indicates when the i -th classical crossing and the j -th critical arc (i.e. $\gamma_{p_j}^j$) of D are incident, the *incidence matrix* of D .

So, if $T_{ij} = [v_i : \gamma_{p_j}^j] \neq 0$, then $M_{ij} = 1$. But the inverse statement is not true at all. For example if $\gamma_{p_j}^j$ is emanating from v_i , passing through it and coming into it, then $[v_i : \gamma_{p_j}^j] = 0$.

Definition 3.3. We say that a classical crossing v_i of D is *cyclic*, if some arc of D is emanating from v_i , passing through it and coming into it.

Thus, if a special diagram D does not have cyclic crossings, then $T_{ij} = [v_i : \gamma_{p_j}^j] \neq 0 \Leftrightarrow M_{ij} \neq 0$. Consequently, if the permanent¹ of matrix M is equal to 1, in other words, there is unique pairing of classical crossings and critical arcs, then $\det T(D) \neq 0$, consequently, D is minimal by Statement 3.2. Thus, we get

¹ $per M := \sum_{\sigma \in S_n} M_{\sigma_1 1} M_{\sigma_2 2} \dots M_{\sigma_n n}$

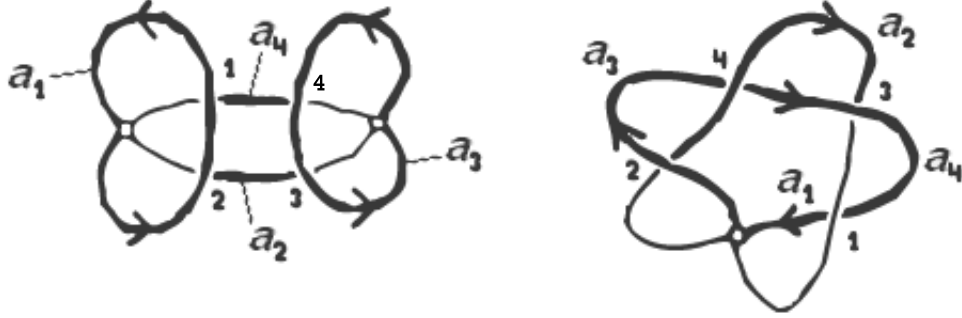


Figure 3: Special diagrams with $\text{per} M > 1$

Theorem 3.1. *If the permanent of the incidence matrix of a special diagram, which does not have cyclic classical crossings, is equal to 1 (i.e. there is unique pairing of classical crossings and critical arcs), then this virtual diagram is minimal with respect to the number of virtual crossings.*

In Figure 3 we illustrate that the sufficient condition for minimality, obtained from Statement 3.2, is much more difficult than the sufficient condition from Theorem 3.1.

For the left diagram in Figure 3, we have

$$\det T = \begin{vmatrix} t-1 & 0 & 0 & -t \\ t^{-1}-1 & 1 & 0 & 0 \\ 0 & -t & t-1 & 0 \\ 0 & 0 & t^{-1}-1 & 1 \end{vmatrix} = (t-1)^2 - (-t)^2(t^{-1}-1)^2 = 0.$$

Consequently, $\deg \zeta < 2$ by Statement 3.2. So, we can not conclude that the diagram is minimal. For the right diagram in Figure 3 we have

$$\det T = \begin{vmatrix} 1 & 0 & 0 & -t^{-1} \\ 0 & 1 & t^{-1}-1 & 0 \\ 0 & -t^{-1} & 0 & t^{-1}-1 \\ 0 & t^{-1}-1 & -t^{-1} & 1 \end{vmatrix} = (t^{-1}-1)t^{-1} - (t^{-1}-1)(-t^{-1} - (t^{-1}-1)^2) = (t^{-1}-1)(t^{-2}+1) = t^{-3} - t^{-2} + t^{-1} - 1 \neq 0. \text{ Thus, the diagram is minimal.}$$

Theorem 3.1 naturally leads to

Definition 3.4. A special diagram, which does not have cyclic classical crossings and has unique pairing of classical crossings and critical arcs, we call an M -diagram.

For detecting nonequivalence of M -diagrams it is convenient to use the following two invariants. Let virtual link L have an M -diagram D . Then by Statement 3.2 and the property of ζ -polynomial we have $\det T(D') = t^l \det T(D)$, where D and D' are M -diagrams of L . But $\det T(D) = \det M(D) t_{1j_1} \dots t_{nj_n}$

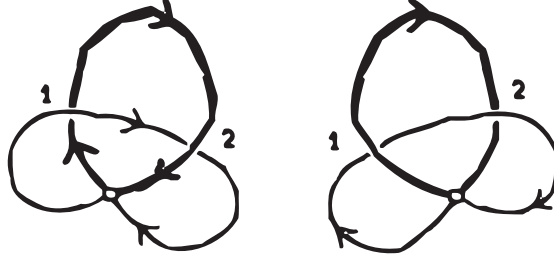


Figure 4: M -knots T and \tilde{T}

(for some j_1, \dots, j_n) can be rewritten in the form $\varepsilon t^\alpha (t-1)^\beta$, where $\varepsilon = \pm 1$, $\beta \geq 0$, $\alpha, \beta \in \mathbb{Z}$, because $t_{ij} = 1, t^{\pm 1} - 1$ or $-t^{\pm 1}$ and $t^{-1} - 1 = (-t^{-1})(t-1)$. Note that numbers ε, α and β quite defined for a product of polynomials $\pm 1, t^{\pm 1} - 1, -t^{\pm 1}$. Thus, numbers ε and β are invariants for virtual link L .

We can formulate the definition of the invariants ε and β in a more geometric way. M -diagram D has unique pairing of classical crossings and critical arcs. In the sense of this pairing, $\beta(D)$ is the number of classical crossings paired with critical arcs which are passing through them. $\varepsilon(D) = \det M(D) (-1)^{x(D)+y(D)}$, where $x(D)$ is the number of negative classical crossings (i.e. their local writhe number is equal to -1) which paired with critical arcs that are passing through them, $y(D)$ is the number of classical crossings paired with critical arcs which are coming into them, $\det M(D) = \pm 1$ characterizes oddness of the pairing of the M -diagram.

Definition 3.5. We say that virtual link is M -link, if can be represented by an M -diagram.

By using the ε -invariant, we prove that all M -knots shown in Figure 4 are different.

Indeed, we have $M(T) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $\det M = -1$, $x = 0$, $y = 1 \Rightarrow \varepsilon(T) = \det M (-1)^x (-1)^y = 1$;

$\varepsilon(\tilde{T}) = \det \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} (-1)^1 (-1)^0 = -1$. The virtual knots shown in Figure 5 are distinguished by the β -invariant.

Indeed, $\beta(Q) = 1$, $\beta(T'_2) = 2$, but ε -invariant does not distinguish between

these knots, because we have $\varepsilon(Q) = \det \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} (-1)^0 (-1)^2 = 1$ and

$\varepsilon(T'_2) = \varepsilon(T)^2 = 1$. Note that the equation $\varepsilon(T'_2) = \varepsilon(T)^2$ holds by the following construction of *special sum* of special diagrams.

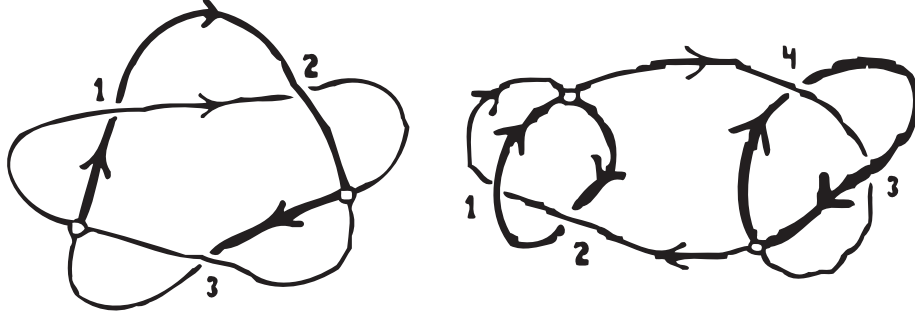


Figure 5: M -knots Q and T'_2

For construction new T - and M -diagrams it is convenient to use following operation of *connected sum of special diagrams*. There are two types of arcs, which do not critical: *pre-critical*, which precede a critical arc on a long arc, and *post-critical*, which follow a critical arc.

A connected sum of special diagrams D_1 and D_2 (or, in other words, only one special diagram $D = D_1 \sqcup D_2$, which is disconnected sum of diagrams D_1 and D_2) is constructed by joining of following parts of arcs which belongs to D . The boundary of the planar graph of D consists of some edges of the graph. Each edge is a part of an arc of D (or the whole arc). We choose a pair of edges in the boundary of D which corresponds to a pair of pre-critical arcs (or post-critical arcs) of $D = D_1 \sqcup D_2$ (in first case we talk about *pre-critical sum* of special diagrams, in the second case we talk about *post-critical sum*) and connect this pair, by using pair of lines which oriented in the opposite manner. We call this connected sum of special virtual diagrams *special*.

It is clear that a special connected sum of special diagrams is special, too. Besides, each critical arc of D remains critical in the special connected sum for D .

One of important characteristics of a special diagram is its T -matrix. So, it is important to compare the T -matrix of a special diagram D and the T -matrix of its special sum \tilde{D} . Let classical crossings and long arcs of diagrams D and \tilde{D} be enumerated correspondingly, i.e. the i -th long arc is emanating from the i -th classical crossing, $i = 1, \dots, n$, and identical classical crossings of D and \tilde{D} have the same numbers. Let b_i and b_j be a pair of pre-critical (or post-critical) arcs of D which take part in the special sum. $\gamma^i \supset b_i$ and $\gamma^j \supset b_j$ denotes long arcs of D . Let $a_i \subset \gamma^i$ and $a_j \subset \gamma^j$ be critical arcs of D . $\tilde{a}_i \subset \gamma^i$, $i = 1, \dots, n$, denote critical and long arcs of \tilde{D} .

So, if the special sum, which is constructed by joining the arcs b_i and b_j , is pre-critical, then $\tilde{a}_j = a_i$, $\tilde{a}_i = a_j$, consequently, $T(D)$ and $T(\tilde{D})$ differ only by the transposition of the i -th and the j -th columns. But if the special sum

is post-critical then $a_i \subset \tilde{\gamma}^i$ and $a_j \subset \tilde{\gamma}^j$, hence $T(D) = T(\tilde{D})$.

Assume the diagram D is composed by two split pieces D_1 and D_2 . Then $T(D) = \begin{pmatrix} T(D_1) & 0 \\ 0 & T(D_2) \end{pmatrix}$ for a convenient enumeration of classical crossings of D . Thus, the above argument yields the following equalities for the connected sum $\tilde{D} = D_1 + D_2$:

$$\det T(D_1 + D_2) = \pm \det T(D) = \pm \det T(D_1) \det T(D_2)$$

$$\text{per} M(D_1 + D_2) = \text{per} M(D) = \text{per} M(D_1) \text{per} M(D_2)$$

Consequently, the T - and M -properties are invariant under the connected sum operation (Note that the connected sum of M -diagrams does not create cyclic crossings, since a set of critical arcs of a connected sum are obtained by the union of the sets of critical arcs of the summands, hence, the critical arc which is incident to a classical crossing three times, can not appear unless it exists in one of the summands).

Thus, the connected sum operation described above allows to construct new M -diagrams from the old ones. Which of them form new M -links? To answer this question, it is useful to understand the behaviour of β and ε under the connected sum operation for M -diagram.

Note that for M -diagrams, the connected sum preserves not only critical arcs, but also the pairings: classical crossing — critical arc. Thus, the invariant β which is equal to the number of classical crossings paired with overpassing critical arcs, satisfies the following equality:

$$\beta(D_1 + D_2) = \beta(D_1) + \beta(D_2)$$

The same argument yields $x(D_1 + D_2) = x(D_1) + x(D_2)$, $y(D_1 + D_2) = y(D_1) + y(D_2)$, where x is the number of negative classical crossings paired with critical arcs going over, and y is the number of classical crossings paired with incoming critical arcs. Besides, $\det M(D_1 + D_2) = -\det M(D_1) \det M(D_2)$, if the connected sum is taken with respect to pre-critical short arcs, and $\det M(D_1 + D_2) = \det M(D_1) \det M(D_2)$, if it is taken with respect to post-critical ones. This yields $\varepsilon(D_1 + D_2) = \det M(D_1 + D_2) (-1)^{x(D_1 + D_2) + y(D_1 + D_2)} = \pm \varepsilon(D_1) \varepsilon(D_2)$, where the sign $-$ or $+$ is chosen according to the connected sum type: we take $-$ for a pre-critical one and $+$ for the post-critical one.

The invariants ε and β and the connected sum construction of M -diagram allow to construct the following four series of M -knots.

1) The Ω -series are shown in Fig. 6.

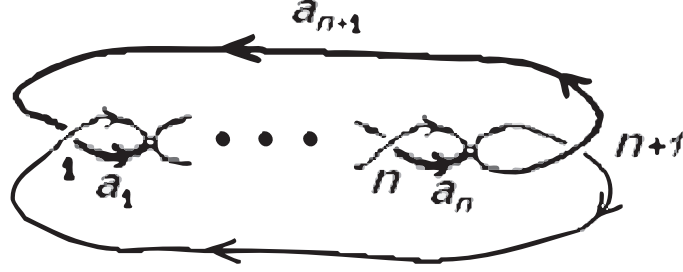


Figure 6: The M -knot Ω_n

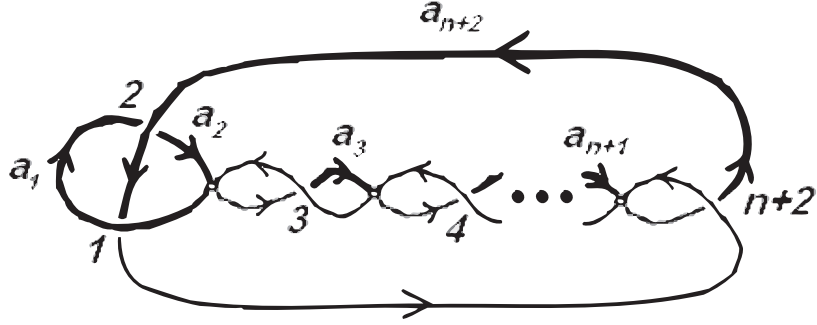


Figure 7: The M -knot W_n

We have $M(\Omega_n) = \begin{pmatrix} 1 & 0 & \dots & 0 & 1 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}$, $\det M = 1$, $x = 1$, $y = 0 \Rightarrow$
 $\varepsilon(\Omega_n) = \det M (-1)^x (-1)^y = -1$. $\beta(\Omega_n) = 1$.

2) The W -series are shown in Fig. 7.

We have $M(W_n) = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 & 1 \\ 1 & 1 & 0 & \dots & 0 & 1 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$, $\det M = 1$, the classical

crossing v_i is paired with the critical arc a_i , $i = 1, \dots, n + 2$, thus $\beta(W_n) = 1$; $x = 0$, $y = 0$, consequently $\varepsilon(W_n) = \det M (-1)^x (-1)^y = 1$.

3) The T -series are obtained from M -knots T and \tilde{T} by using connected sums. This connected sum is represented schematically by a graph. The vertices of the graph correspond to virtual diagrams (summands, in our case T and

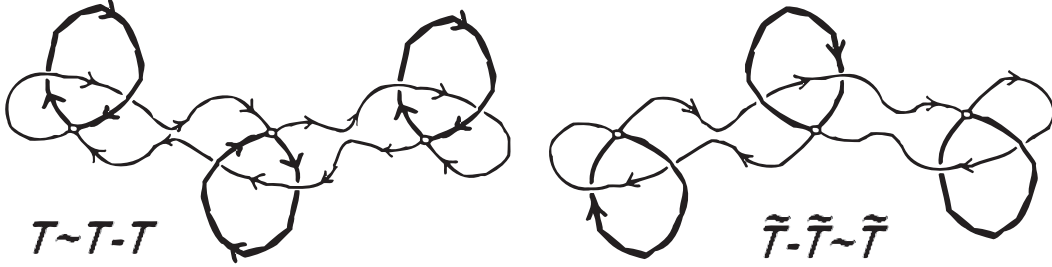


Figure 8: Examples of knots: T -series and their schemes

\tilde{T}), and the edges correspond to the “tubes” $S^0 \times I$ for the connected sums; these edges are of two types: pre-critical and post-critical, which is graphically represented by a wave line or by a straight line, respectively, see Fig. 8.

Clearly, there are virtual diagrams with schemes $\underbrace{T \sim T - T \sim T - \dots - (\sim)T}_n$ and $\underbrace{\tilde{T} - \tilde{T} \sim \tilde{T} - \tilde{T} \sim \dots - (\sim)\tilde{T}}_n$. Denote these diagrams by T_n and \tilde{T}_n , respectively.

According to the above property of connected sums of M -diagrams, we have $\varepsilon(A \sim B) = -\varepsilon(A)\varepsilon(B)$, $\varepsilon(A - B) = \varepsilon(A)\varepsilon(B)$. By $\varepsilon(T) = 1$, $\varepsilon(\tilde{T}) = -1$, this yields $\varepsilon(T_n) = 1 \Leftrightarrow n \equiv_4 0$ or 1 , and $\varepsilon(\tilde{T}_n) = 1 \Leftrightarrow n \equiv_4 2$ or 3 , thus $T_n \approx \tilde{T}_n$ for all n .

4) The Q -series are constructed by using connected sums of the diagram Q shown in Fig. 5. By applying the connected sum operation to Q , we can attach its parallel copies in four directions, see Fig. 9.

As we see, in order to get a one-component link, it is not necessary to take a tree for a scheme of a Q -knot.

To distinguish the knots from Q -series, let us use the ε -invariant. Since $\varepsilon(Q) = 1$, the value of ε on a connected sum of r copies of Q is equal to $+1$ if and only if the number of wave lines in the scheme is even. Thus, for $r \geq 2$ there are at least two knots in the Q -series, for which the minimal virtual crossing number is equal to $2r$, since in the scheme of connected sum we may change the parity of the number of wave lines.

The information about these series is collected in the following table.

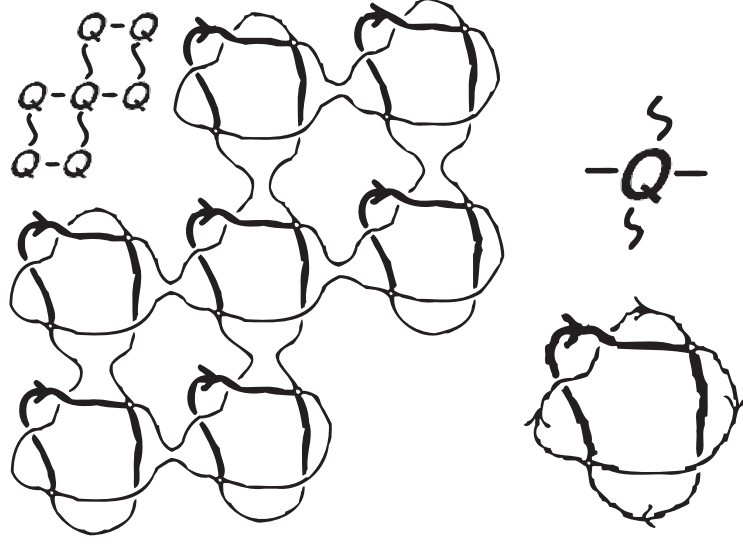


Figure 9: Constructing the Q -series of knots

	Ω_n	W_n	T_n, \tilde{T}_n	$Q_n, n = 2r$
ε	-1	1	± 1	± 1
β	1	1	n	$\frac{n}{2}$
The number of classical crossings	$n+1$	$n+2$	$2n$	$\frac{3}{2}n$

Hence, for $n \geq 2$ the knots $\Omega_n, W_n, T_n, \tilde{T}_n$ are pairwise distinct, and for even $n \geq 4$ the knots $\Omega_n, W_n, T_n, \tilde{T}_n, Q'_n, Q''_n$ are pairwise distinct, where Q'_n, Q''_n are any two Q -knots with minimal virtual crossing number n . It is easy to check that for $n = 1$ the series listed above consist of the following three knots: $T_1, \tilde{T}_1 \sim \Omega_1, W_1$.

4 Discussion and open questions.

It is known that classical links embed into virtual links. More precisely, if two classical diagrams are connected by a chain of virtual Reidemeister moves, then they can be connected by a chain of classical Reidemeister moves (see [GPV]).

There is an analogous question between the relation of virtual links generated by M -diagrams and all virtual links.

If one wants to classify some objects, say, virtual links, it would be natural to split the object of classification into some “layers”: say, classical links (those having no virtual crossings at all), links with one virtual crossing, links with two virtual crossing etc. Possibly, one may tackle this problem by using a subset of links represented by M -diagrams.

Therefore, we list some open questions concerning M -property.

CONJECTURE 1. *M -property of proper virtual diagrams is stable with respect to the virtual Reidemeister moves $\Omega_1, \Omega_2, \Omega_3, \Omega', \Omega'_3$.*

CONJECTURE 2. *If virtual diagrams are equivalent and satisfy M -property, then they are connected by Reidemeister moves $\Omega_1, \Omega_2, \Omega_3, \Omega', \Omega'_3$.*

Moreover, it can be formulated more strong

CONJECTURE 3. *If virtual diagrams D and D' are equivalent, D satisfies M -property and D' is minimal, then D' satisfies M -property.*

Finally, we address the following two questions.

1) *Is the number of nonequivalent M -diagrams which have the same fixed number of virtual crossings, finite?*

2) *Is it true that the number of classical crossing for a one-component M -diagram greater than the number of virtual crossings?*

References

- [BF] A. Bartholemew, R. Fenn (2003), Quaternionic invariants of virtual knots and links, *Journal of Knot Theory and Its Ramifications*, **17** (2), 2008 pp. 231–251
- [DK] H. Dye, L.H. Kauffman, Virtual Crossing Number and the arrow Polynomial, Arxiv.Math/GT. 0810:3858.
- [FKM] R. A. Fenn, L. H. Kauffman, V. O. Manturov, Virtual knots - unsolved problems (2006), *Fundamenta Mathematicae*, N. 188, pp. 293–323.
- [GPV] M. Goussarov, M. Polyak, and O. Viro, Finite type invariants of classical and virtual knots, *Topology*. 2000. V. 39, pp. 1045–1068.
- [Jon] V.F.R. Jones, A polynomial invariant for links via Neumann algebras (1985), *Bull. Amer. Math. Soc.*, **129**, pp. 103–112.
- [Ka1] L.H. Kauffman, Virtual knot theory, *Eur. J. Combinatorics*. 1999. V. 20, N. 7, pp. 662–690.

- [Ka2] L. H. Kauffman, State Models and the Jones Polynomial (1987), *Topology*, **26**, pp. 395–407.
- [KM] L.H.Kauffman, V.O.Manturov, Virtual Knots and Links, Proceedings of the Steklov Inst, 2006, Vol. 252, N.1, pp. 104-121.
- [KR] L.H.Kauffman, D.Radford (2002), Bi-oriented quantum algebras and a generalized Alexander polynomial for virtual links, *AMS Contemp. Math.*, **318**, pp. 113-140.
- [Kup] G. Kuperberg, What is a Virtual Link? (2002), www.arXiv.org, math-GT/ 0208039, *Algebraic and Geometric Topology*, 2003, **3**, 5 87-591.
- [Ma1] V.O.Manturov, An Invariant 2-variable polynomial for virtual links (2002), (Russian Math. Surveys), **57**, No.5, P.141-142.
- [Ma2] V.O.Manturov, The Khovanov complex and minimal knot diagrams (2006), *Doklady Mathematics*, **73**, N.1, P. 46-48.
- [Ma3] V. O. Manturov, *Teoriya Uzlov* (Knot Theory), (Moscow-Izhevsk, RCD), 2005 (512 pp).
- [Miy] Miyazawa, Y (2006), Magnetic Graphs and an Invariant for virtual links, *J. of Knot Theory and Its Ramifications*, **15** (10), pp.1319-1334.
- [Mur] K. Murasugi, The Jones Polynomial and Classical Conjectures in Knot Theory, *Topology*, **26**, 1987, pp. 187-194.
- [Saw] J. Sawollek (2002), On Alexander-Conway Polynomials for Virtual Knots and Links, *J. Knot Theory and Its Ramifications*, **12** (6), pp.767-779.
- [SW] D.Silver and S.Williams (2001), Alexander Groups and Virtual Links, *J. of Knot Theory and Its Ramifications*, **10** (1), pp. 151-160.
- [Tep] E.V.Teplyakov, On Roots of the ξ -polynomial (2005), *Journal of Mathematical Sciences*, **128** (4), P. 3107-3113.
- [Thi] M. Thistlethwaite, A spanning tree expansion for the Jones polynomial, *Topology*, **26**, 1987, pp. 297-309.